Collatz Proof   Dec 9, 2024   Dave Cromley

Why should Anyone look at THIS proof?

This approach uses graph theory to prove that the Collatz graph is a tree and includes all positive integers. Previous attempts have produced graphs that were connected and acyclic (therefore a tree) but were not proven to contain all numbers.

What is new in this approach is using sequences of even numbers and sequences of odd numbers in constructing the graph. With these sequences, number completeness can be proven.

Figure 2 shows the graph with the sequences of even numbers being horizontal and the sequences of odd numbers dropping down.

If you are familiar with graph theory and trees, I invite you to review.

The Collatz Conjecture

The Collatz conjecture asks whether repeating two simple arithmetic operations will eventually transform every positive integer into 1. It concerns sequences of integers in which each term is obtained from the previous term as follows: if a term N is even, the next term is N/2. If a term N is odd, the next term is (3N+1)/2. The conjecture is that these sequences always reach 1, no matter which positive integer is chosen to start the sequence.

Abstract of proof

We construct a graph with a root of 1. Nodes are connected using the inverse Collatz functions. The degenerate 2-1 connection that causes a loop is excluded.

The graph is proven to be connected and acyclic and therefore a tree.

The nodes are considered in sequences of even and odd numbers.

Limbs are sequences of even numbers generated from an odd number Q.
Every odd number Q generates a Limb.
Every even number appears in exactly 1 Limb.

Twigs are sequences of odd numbers generated from an even number F.
Some (1/3) of even numbers F generate a Twig.
Every odd number appears in exactly 1 Twig.

From this, the tree is proven to contain all positive integers.

Given any number N, we can descend the tree to the root 1 by using the Collatz functions.

Section A: Notations and definitions

This proof uses the "shortcut" form of the Collatz functions.

  Ref: https://en.wikipedia.org/wiki/Collatz_conjecture

"numbers" are positive integers.

For numbers A and B, the expression A (mod B) is shortened to AmodB for readability.

"even 2mod3" is equivalent to "2mod6" but is more meaningful for this paper.

N is any number.

M is any 2mod3 number (2, 5, 8, …).

E is any even number.

F is any even 2mod3 number (2, 8, 14, …).

Q is any odd number.

Section B: The inverse Collatz (shortcut) functions and lemmas

The shortcut Collatz functions:

For even numbers, N' = N/2.

For odd numbers, N' = (3N+1)/2.

In constructing the graph, we use the inverse functions:

Inverse of N/2: N' = 2N.

A child connected with this function is called a D-child (D for Double).

Inverse of (3N+1)/2: N' = (2N-1)/3.

A child connected with this function is called an S-child (S for Smaller).

Lemma #1 - A D-child of a node is even.

This is immediate from the definition of the function N' = 2N.

Lemma #2 - An S-child of a node is odd.

Only a 2mod3 number M can have an S-child. Then M = 3n+2 for some n.

Applying the N'=(2N-1)/3 function to M gives 2n+1, which is odd. QED.

Section C: Branches - Limbs

For every odd number Q, we define a "Limb" to be the (infinite) sequence of even numbers created by repeatedly connecting a D-child using N' = 2N.

The ith term in the Limb sequence Q is:

ith term = $2^i \times Q$ for $i \geq 1$

Every even number F is in exactly 1 unique Limb, and that Limb Q is:

Q (odd) = $F / 2^K$ where K is the maximum power of 2 that divides F

Here are some Limbs (sequences of even numbers generated from an odd number Q):

Q=1: (2 4 8 ...)
Q=3: (6 12 24 ...)
Q=5: (10 20 40 ...)
Q=7: (14 28 56 ...)

Section D: Branches - Twigs

For every even 2mod3 number F, we define a "Twig" to be the (finite) sequence of odd number(s) created by repeatedly connecting an S-child using N' = (2N-1)/3 until N' is not 2mod3.

The ith term in the Twig sequence F is:

*ith term = $(2/3)^i \times (E+1) - 1$ for $1 \leq i \leq K$

where K = the highest power of 3 that divides (E+1).

The length of the sequence is K.

*The ith term statement is easily proven by induction.

Every odd number Q is in exactly 1 unique Twig, and that Twig F is:

F (even) = $(3/2)^K \times (Q+1) - 1$ where K is the maximum power of 2 that divides Q+1

This equation is derived from ith term statement.

Here are some Twigs (sequences of odd numbers generated by an even 2mod3 number F):

F=2: (1) (excluded -- see below)
F=8: (5 3)
F=14: (9)
F=20: (13)
F=26: (17 11 7)

The graph excludes the degenerate Twig F=2 which causes the 2-1-2 loop.

Section E: The Graph

The root of the graph is the number 1.

Every odd number Q connects to its associated Limb (sequence of even numbers).

Every even 2mod3 number F connects to its associated Twig (sequence of odd numbers).

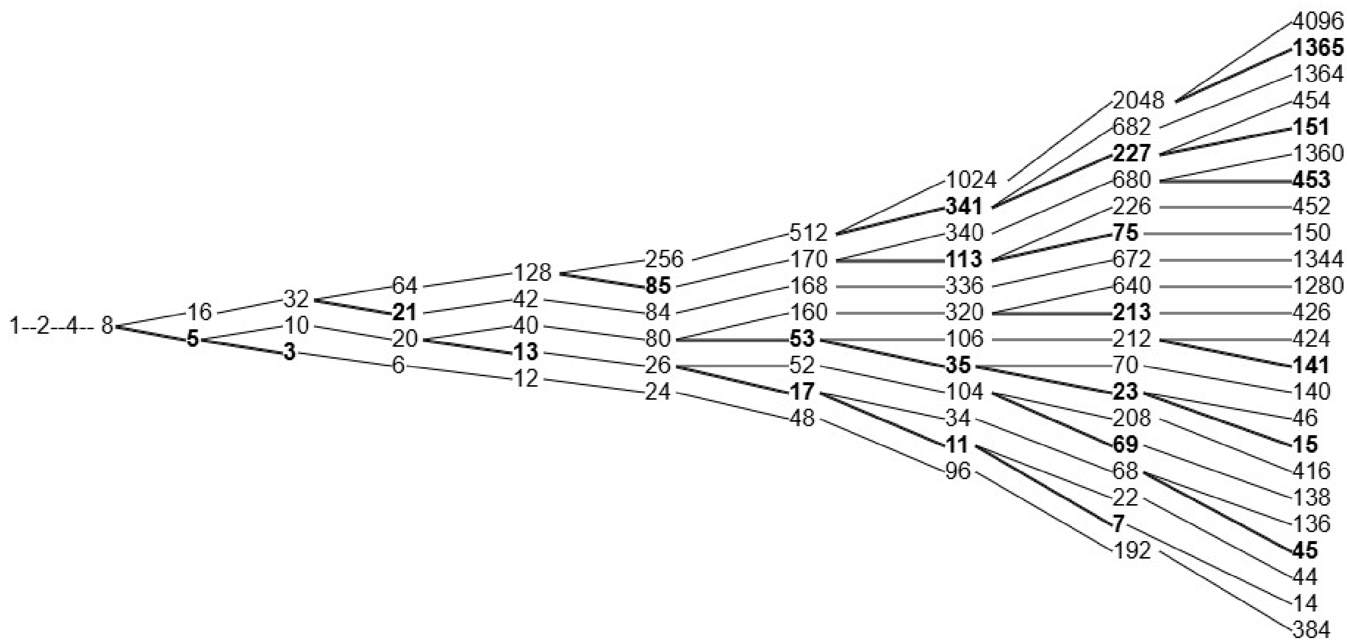

Figure 1 - The Collatz tree to a height of 12. The Twigs are bolded.

Section F: A second view of the Collatz graph.

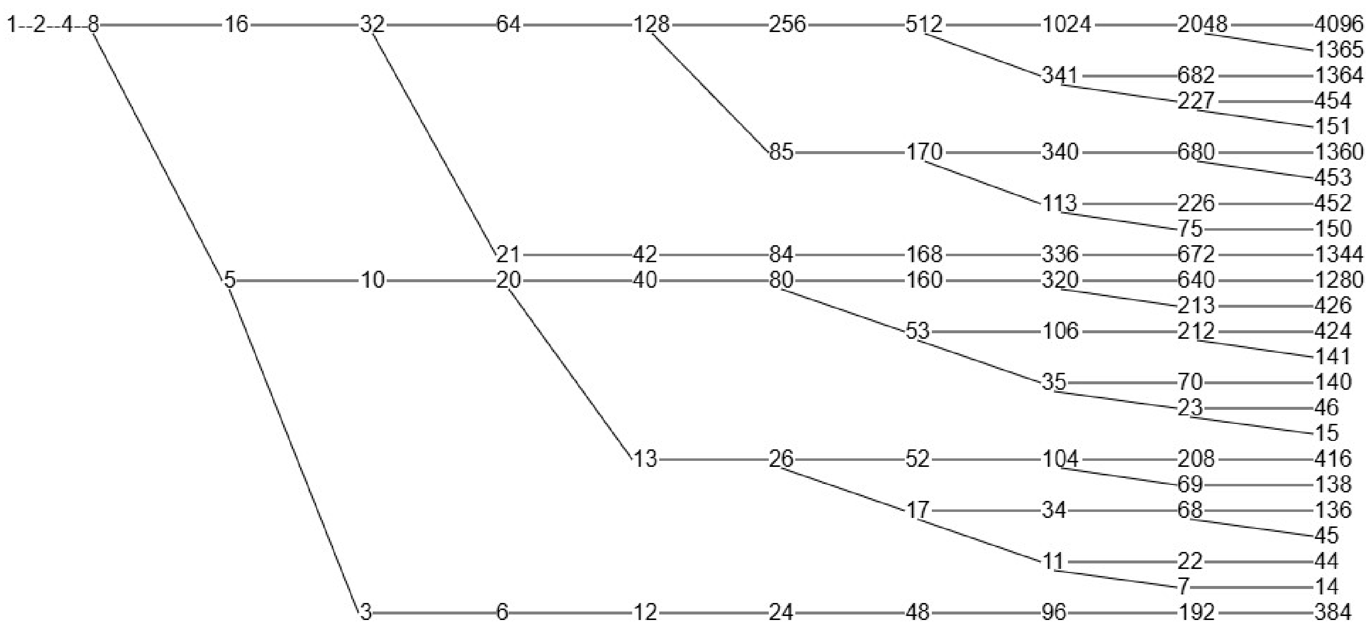

Figure 2 - is also the Collatz graph to a height of 12, but it emphasizes the Limbs and Twigs.
Limbs of even numbers go off to the right.
Twigs of odd numbers drop down.

==== The Proof ====

Prove that the Graph is a tree

- Prove the graph is connected. By construction:

  1. The root is 1
  2. At this point all leafs are odd numbers that will connect to Limbs.
  3. For all leafs, connect the corresponding Limb.
  4. At this point, all leafs are even numbers, some of which are 2mod6 which will connect to Twigs.
  5. For all 2mod6 leafs, connect the corresponding Twig.
  6. Go to Step 2.

  Therefore the graph is connected.

- Prove the graph is acyclic. By contradiction:

  There are only 2 types of child nodes, the D-child and the S-child.

  A node cannot be a D-child from 2 different parents because the inverse functions connecting back would make both parents the same.

  Similarly, a node cannot be an S-child from 2 different parents.

  A node cannot be a D-child from one parent and be an S-child from another because, from lemmas #1 and #2, being a D-child makes the node even and being an S-child makes the node odd.

  Therefore every node except the root has exactly 1 parent proving the graph acyclic.

- Being connected and acyclic, the graph is proven to be a tree.
- Prove the tree contains all numbers.

  From Section C, every even number is in a Limb sequence.

  From Section D, every odd number is in a Twig sequence.

  By tree connectivity, all node sequences are in the tree.

  Therefore all numbers are in the tree.

QED

- We now have the graph being a tree with all numbers
- The path from the root 1 to any N is built with the inverse Collatz functions.
- Therefore the path can be followed from N to the root 1 using the Collatz functions themselves.

Dave Cromley

dave@dbarc.net

307.286.7513

Appendix A - creating a Collatz tree to a height of H.

As always, the Collatz tree is created using the inverse Collatz functions, The terms Limb and Twig are used here, but the arithmetic is the same as it has been. Figures 1 and 2 were created using these steps:

Step 1: The graph starts with the root node of 1 (a leaf).

Step 2: For every leaf, we connect a D-child using the function

N' = 2N. This accomplishes the following:

- For odd nodes, a new Limb is started.
- For even nodes, the Limb is extended.

Step 3: For every leaf that is 2mod3 except 2, we connect an S-child using the function

N' = (2N-1)/3 This accomplishes the following:

(2mod3 nodes are either 2mod6 or 5mod6):

- For 2mod6 nodes, a new Twig is started.
- For 5mod6 nodes, the Twig is extended.

Step 4: Repeat steps 2 and 3 until the desired height H is reached. If one has a specific N in mind, the Height of N can be found by applying the Collatz functions to N until the root 1 is reached and counting the steps (a finite number).

After every step 3, the height of all leafs are the same, which is the number of times we have performed steps 2 and 3.

Appendix B - A third view of the Collatz tree.

The following 3 figures show the tree in a different view.

The nodes are equally spaced from left to right.

The Limb edges connecting to even numbers are shown above the midline.

The Twig edges connecting to odd numbers are shown below the midline.

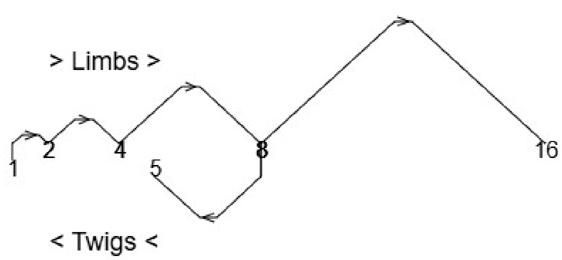

Figure B1 - The Collatz tree to a height of 4,
The 2 leafs are 16 and 5.
The 2 paths are (1 2 4 8 16) and (1 2 4 8 5)

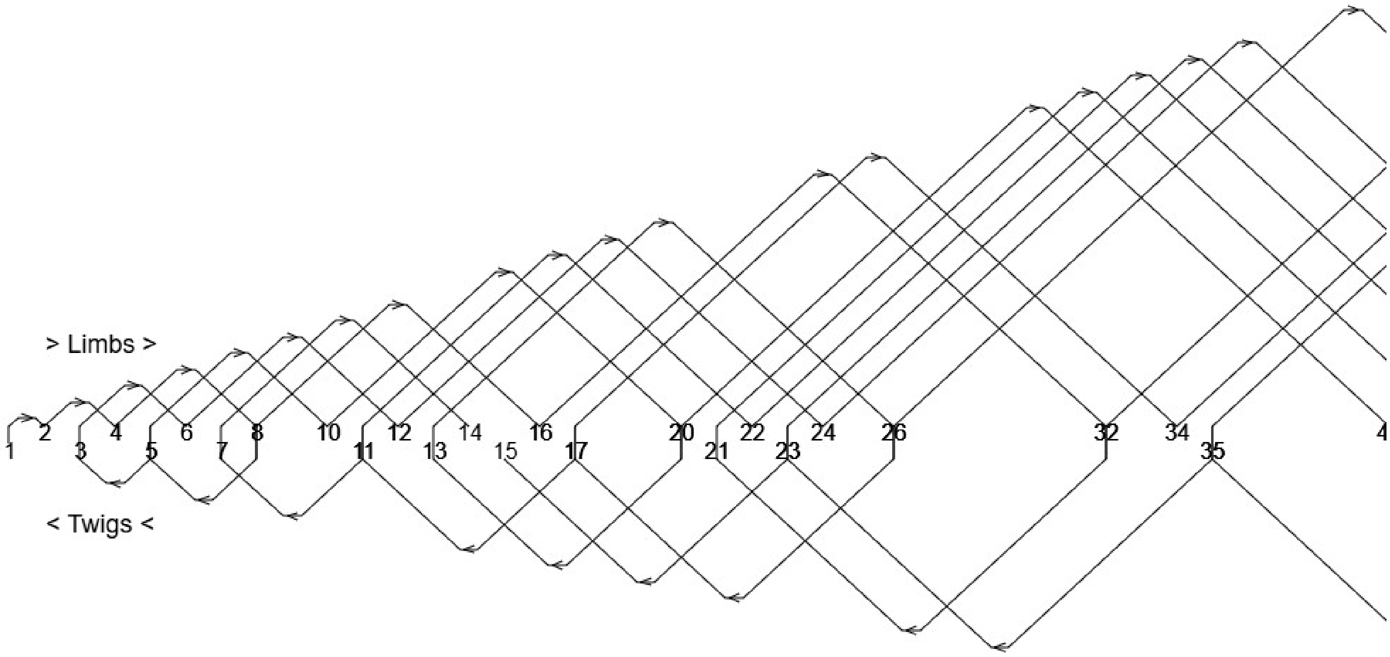

Figure B2 - The Collatz tree to a height of 12. This is the same height as figures 1 and 2.
Most of the nodes are off the figure.

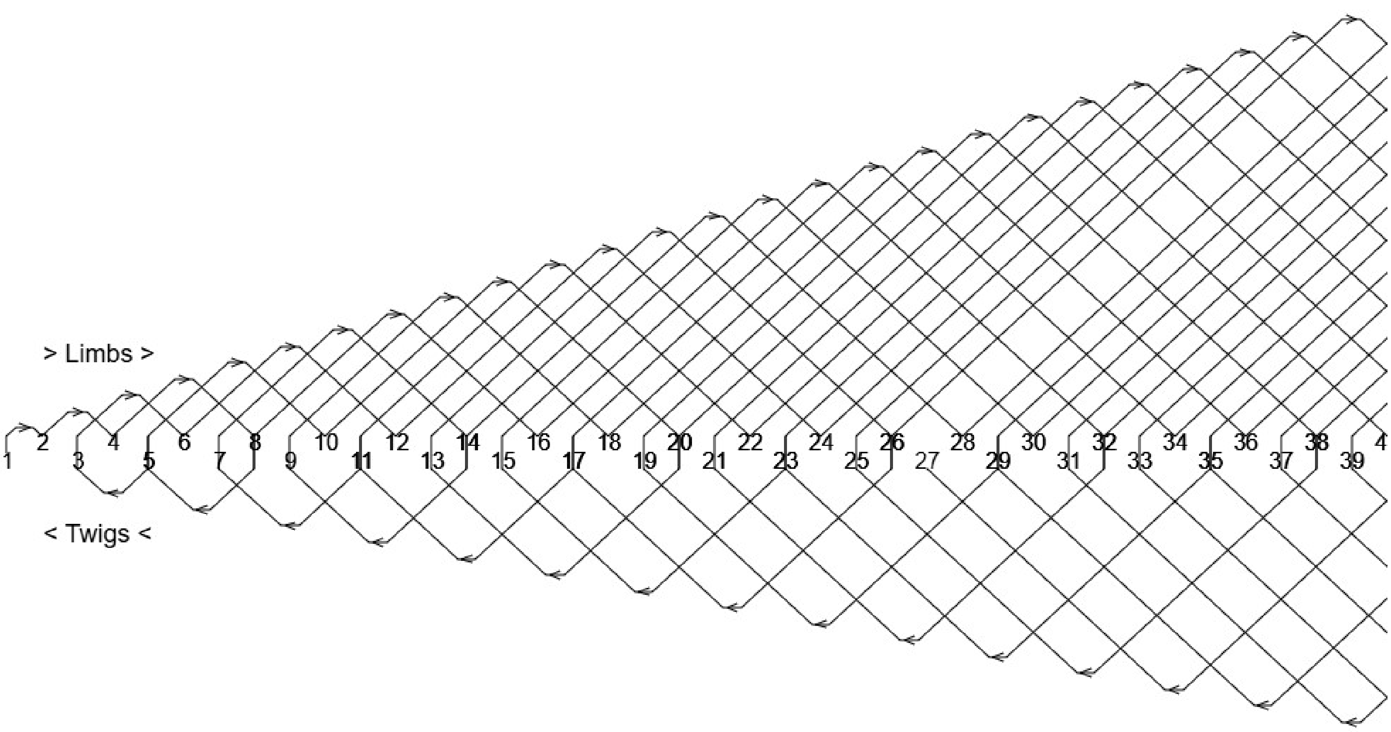

Figure B3 - The Collatz tree to a height of 70.
The next level would connect 27 to 54.
Numerically, the highest node in the path to 27 is 4,416 at a height of 25.
This form of the tree was the inspiration for the "Limbs" and "Twigs".